CHAPTER 6

Perspective on "The Child's Thought and Geometry"

As mathematics educators in Montessori secondary schools in the Netherlands, Pierre and Dina van Hiele were disappointed with students' low-level knowledge of geometry and intrigued by the phenomenon of teachers' and students' "failure to communicate." In 1957, at the University of Utrecht, the van Hieles completed companion doctoral dissertations. Pierre van Hiele formulated a system of levels of thinking in geometry. Dina van Hiele-Geldof focused on a teaching experiment to raise students' thought levels.

In *The Child's Thought and Geometry*, Pierre van Hiele describes their influential theory of levels of geometry thinking. According to this theory, students progress through levels of thought in geometry. The theory is based on several assumptions. First, geometry learning is a discontinuous process characterized by qualitatively different levels of thinking. Such levels progress from a Gestalt-like visual level through increasingly sophisticated levels of description, analysis, abstraction, and proof. Second, these levels are sequential, invariant, and hierarchical. Progress is dependent on instruction, not age. Teachers can "reduce" subject matter to a lower level, leading to rote memorization, but students cannot bypass levels and achieve understanding. The latter requires working through certain "phases" of instruction. Third, concepts implicitly understood at one level become explicitly understood at the next level. Fourth, each level has its own language; teachers unaware of these features of students' learning can easily misinterpret students' understanding of geometric ideas.

This theory, then, builds on the constructivism and geometric studies of Piaget but also heads in new directions. The theory emphasizes mathematical content at the core of research and practice. It clearly attributes students' development to the teaching-learning process.

Theories are useful if they are used—and contested, attacked, and modified. By this criterion, van Hiele's theory is a useful theory. It has precipitated a large body of work using, evaluating, and modifying the theory itself. It has deepened and expanded research in the learning and teaching of geometry, not only affecting research on such palpable topics as geometry figures and proof but also serving as the theoretical backbone on research in a wide range of related topics, including assessment, educational technology and the use of manipulatives, students with special needs, analyses of textbooks, and curriculum development. The theory has also served to inspire creative thinking in educational arenas not connected with geometry, such as professional development.

Van Hiele's theory gave educators and researchers a model that promoted the understanding of important, conceptually based levels of thinking; that emphasized the primacy of mathematical content; and that was intrinsically connected with educational as well as psychological concerns. It is also a model of synergistic connections among theory, research, the practice of teaching, and students' thinking and learning.

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The Child's Thought and Geometry

By P. M. van Hiele

The art of teaching is a meeting of three elements: teacher, student, and subject matter. Since it is very difficulty to keep all of these things in view at the same time, one has a tendency to neglect one of them, which gives an incorrect view of the situation. Because if one neglects subject matter, one only sees the relationship between teacher and student; if one loses sight of the student, then one only sees the structure of the subject matter. Sometimes one does not sufficiently realize that the teacher is there to direct the student's studies.

Nevertheless let us acknowledge for the sake of argument that one should take into account the three aspects mentioned above without omitting any of them. There remains nonetheless a great danger and it appears to me that it has not been sufficiently recognized. The difficulty which arises is that the subject matter as met, by the student is of a completely different structure from that known by the teacher.

If we agree that the aim of our teaching is that the student should know how to prove theorems, it is highly improbable that the student's thought aims directly towards this goal. Improbable, because the student will not be able to grasp, in its intrinsic sense, the idea of *proving a theorem.* In fact, if he had this idea, he would not have the need to learn it. Understanding mathematics comes down to this: knowing the relationships between theorems that one studies. As soon as one understands the meaning of these theorems, one knows their relationships at the same time.

All this is very simple and shows us clearly why mathematics is so difficult for students. The teacher knows the relationships between the theorems, but he knows them in a different way than the student. His explanation of these relationships does not suffice to make them intelligible to the student. What the student must understand in the first place is that there are such things as theorems. This is all that one can expect from a beginning student. The following example will illustrate what I mean.

A teacher wants to teach plane geometry to beginning students. He uses symmetry with respect to a straight line, in order to teach them the relationships between equality of segments or angles, perpendicularity, etc. He teaches them that the points on the axis of symmetry are invariant, that symmetric segments have the same length, that symmetric lines intersect on the axis of symmetry. In order to see if the students have understood what he has taught, he gives them the following problem: "Let *ABC* be a triangle for which the extensions of the sides meet the line L. Construct the symmetrical triangle with respect to L." The teacher imagines the following solution: "The lines *AB* and *AC* meet the axis L in two points that we call *P* and *Q*. These points are invariant under symmetry. Then the distances *AP* and *AQ* are invariant, so that one can construct the symmetrical points A′. In the same way one finds the points B′ and C′."

All of this reasoning, this whole way of conceiving the material, is the reasoning of a teacher who knows all the relationships. The student is completely incapable of developing a similar process of thought without the teacher's help. The teacher has used the fact that the lengths of symmetrical segments are the same as the basis for his argument. Such a technique is meaningless for the students because they have not yet seen a counterexample; they have not yet seen transformations which change the length of segments.

But there is a more important reason for us to oppose this method of teaching of which we have given an example: it requires students to reason with the help of a system of relations between ideas whose

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meanings they do not even know. It is a matter of "points," "axis of symmetry," "segments," "to meet," "invariant," "to change length," "triangle," "extension." Obviously, the teacher has explained these expressions, he has shown points and segments, he has demonstrated at the blackboard what is meant by extending a segment. Possibly he has asked the students to formulate the definition of vertical angles. It is even possible that the definition was found to be not quite correct and that as result he showed this by means of a counterexample. One must realize however that it is the teacher who is giving the counterexample. The students would fail because to be in a position to give a counterexample one must have a system of relations at one's disposal, and they do not have one.

I hope that the thought that I have just presented to you will have clearly shown that the teacher reasons by means of a system of relations that he alone possesses. Starting with this system, he explains the mathematical relations that the students end up manipulating by rote. Or else the student learns by rote to operate with these relations that he does not understand, and of which he has not seen the origin.

At first glance things seem to be in order: the students will end up having at their disposal the same system as the teacher. Is this not the proper goal of the teaching of mathematics, namely: the possession of a system of relations identical for all those that use it, appropriate to express arguments, a system in which the relations are linked in a logical and deductive fashion?

Let us not be too optimistic. First of all, a system of relations structured in this way is not based on the sensory experiences of the student. Though it is possible that the system of relationships itself has inspired some experiences on the part of the student, the mathematical experiences that the student has been able to have are based only on the system imposed by the teacher. This system, imposed and not understood, forms the base of his reasoning. As one knows, a system of relations which is not based on prior experience has the potential of being forgotten in a short time.

Therefore the system of relations is an independent construction having no rapport with other experiences of the child. This means that the student knows only what has been taught to him and what has been deduced from it. He has not learned to establish the connections between the system and the sensory world. He will not know how to apply what he has learned in a new situation.

Finally, the student has learned to apply a system of relations that has been offered to him ready-made, he has learned to apply it in certain situations specifically designed for it. But he has not learned how to construct such a system himself in a domain which is still unstructured. If, on the other hand, we were to succeed in ensuring as a result of our teaching that the students are capable of constructing for themselves a deductive relational system in a new domain, we would have produced the optimal mathematical training.

In general, the teacher and the students speak a very different language. We can express this by saying: they think on different levels. Analysis of geometry indicates about five different levels.

At the *Base Level* (Level 0) of geometry, figures are judged by their appearance. A child recognizes a rectangle by its form and a rectangle seems different to him than a square. When one has shown a six-year-old child what a rhombus is, what a rectangle is, what a square is, what a parallelogram is, he is capable of reproducing these figures without error on a geoboard of Gattagno, even in difficult arrangements. We have used the geoboard in our research so that the child will not be bothered by the difficulties resulting from drawing figures. At the base level, a child does not recognize a parallelogram in the shape of a rhombus. At this level, the rhombus is not a parallelogram, the rhombus seems to him a completely different thing.

At the *First Level* of geometry, the figures are bearers of their properties. That a figure is a rectangle means that it has four right angles, diagonals are equal, and opposite sides are equal. Figures are recognized by their properties. If one tells us that the figure drawn on a blackboard has four right angles, it is a rectangle even if the figure is drawn badly. But at this level properties are not yet ordered, so that a square is not necessarily identified as being a rectangle.

At the *Second Level*, properties are ordered. They are deduced one from another: one property precedes or follows another property. At this level the intrinsic meaning of deduction is not understood by the students. The square is recognized as being a rectangle because at this level definitions of figure come into play.

At the *Third Level*, thinking is concerned with the meaning of deduction, with the converse of a theorem, with axioms, with necessary and sufficient conditions.

One can probably distinguish five levels of thought in geometry. This number is moreover of little importance in understanding what a level of thought is.

These levels—as we have said—are inherent in the elaboration of thought; they are independent of the method of teaching used. It is possible, however, that certain methods of teaching do not permit attainthe higher levels, so that the methods of [illegible] used at these levels remain inaccessible to students. The following points can contribute to a [illegible]tion of levels of thought:

a. At each level there appears in an extrinsic way that which was intrinsic at the preceding level. At the base level, figures were in fact also determined by their properties, but someone thinking at this level is not aware of these properties.

b. Each level has its own linguistic symbols and its own system of relations connecting these signs. A relation which is "correct" at one level can reveal itself to be incorrect at another. Think, for example of the relation between a rectangle and a square. Numerous linguistic symbols appear at two successive levels; moreover they establish a liaison between the various levels and assume continuity of thought in this discontinuous domain. But their meaning is different: it becomes manifest by other relations among these symbols.

c. Two people who reason at two different levels cannot understand each other. This is what often happens between teacher and student. Neither of them can manage to follow the thought process of the other and their dialogue can only proceed if the teacher tries to form for himself an idea of the students' thinking and to conform to it. Some teachers make a presentation at their own level while asking students to reply to their questions. In fact, it is nothing but a monologue, for the teacher is inclined to consider all the answers which do not belong to his system of relations as stupid or misplaced. A true dialogue must be established at the level of the students. For this to happen, the teacher must often, after class, ask himself about the responses of his students and strive to understand their meaning.

d. The maturation which leads to a higher level happens in a special way. Several stages can be revealed in it (this maturation must be considered above all as a process of apprenticeship and not as a ripening of a biological sort). It is thus possible and desirable that the teacher aids and accelerates it. The aim of the art of teaching is precisely to face the question of knowing how these phases are passed through, and how help can effectively be given to the student.

Let us now examine the *phases which*, in the process of apprenticeship, *lead to a higher level of thought.*

The *first phase* is one of *inquiry:* the student learns to know the field under investigation by means of the material which is presented to him. This material leads him to discover a certain structure. One could say that the basis of human knowledge consists of this: mankind is characterized by the revelation of structure in any material, however disorganized it may be, and this structure is experienced in the same way by several people, which results in a conversation that they can have about this subject.

In the *second phase*, that of *directed orientation*, the student explores the field of investigation by means of the material. He already knows in what direction the study is directed; the material is chosen in such a way that the characteristic structures appear to him gradually.

In the course of the *third phase, explicitation* takes place. Acquired experience is linked to exact linguistic symbols and the students learn to express their opinions about the structures observed during discussions in class. The teacher takes care that these discussions use the habitual terms. It is during this third phase that the system of relations is partially formed.

The *fourth phase* is that of *free orientation.* The field of investigation is for the most part known, but the student must still be able to find his way there rapidly. This is brought about by giving tasks which can be completed in different ways. All sorts of signposts are placed in the field of investigation: they show the path towards symbols.

The *fifth phase* is that of *integration:* the student has oriented himself, but he must still acquire an overview of all the methods which are at his disposal. Thus he tries to condense into one whole the domain that his thought has explored. At this point, the teacher can aid this work by furnishing global surveys. It is important that these surveys do not present anything new to the student; they must only be a summary of what the student already knows.

At the close of this fifth phase a new level of thought is attained. The student has at his disposal a system of relations which are related to the whole of the domain explored. This new domain of thought, which has acquired its own intuition, is substituted for the previous domain of thought which had a completely different intuition.

The objectivity of mathematics rests on the fact that new systems of relations are agreed on by different people. The new symbols are linked by the same relations among many people. If one decides that the goal of education should be the uniqueness of the relational system, one could restrict oneself to having that learned. And the student would seem to understand the reasoning perfectly, for it would result in correct conclusions based or his relational system. But that is not to say that he would attach to it the same significance as his questioner. This significance cannot be disentangled solely from the language used, it depends also on the experiences which led to the formation of the relational system, that is, it depends on what happened at a lower level of thought.

If one does not take the content of the symbols into consideration, but only their relations, one could say that from a mathematical point of view, everything is perfect. The student is capable of handling the relational system of deduction without mistakes. But from the pedagogical and didactic point of view, and from the social point of view, one has wronged the student! One has committed a pedagogical error because one has stolen from the student an occasion to realize his creative potential. From the didactic point of view, one has neglected to let the student discover how to explore new domains of thought by himself. Finally, one has wronged society because one has provided the student with a tool which he can handle only in situations which he has studied.

The theory of levels of thought leads to the following important conclusions.

1. One has been able to see that the levels of thought are inherent in thought itself; thus they are not only the concern of those who occupy themselves with didactics. The levels of thought have, for example, a certain importance for mathematics itself. One can only express oneself clearly in mathematics when one uses symbols belonging to one's own level. If one manipulates functions, it is of little importance that they are defined by the expression $f(x)$ or by the equation $y = f(x)$. One learns to know the function while using it and out of this activity flows the content of the notion of function. If one asks oneself, at a higher level, the questions of what a function is, of *what one has really done*, one will arrive at the conclusion that it is a pairing of elements x and of elements $f(x)$. The function is defined neither by $f(x)$, nor by $y = f(x)$, but rather by the symbol for the pairing which one can represent, if one wants, by f. Error results from trying to give a definition at a lower level of thought, from exploiting a structure contained implicitly in an activity before it has become sufficiently familiar. Because this attempt is doomed to failure, one limits oneself to representing either the result of this action, $f(x)$, or else the action itself, $y = f(x)$. The mistake is not only a didactic one, but also theoretical. (This example is drawn from a conversation with Professor Freudenthal.)

   One makes an analogous error when one tries to construct a system of axioms using symbols which belong to a level of thought which is too low. Systems of axioms belong to the fourth level where in fact one no longer asks the question: what are points, lines, surfaces, etc.? At this fourth level, figures are defined only by symbols bound by relations. To find their appropriate content, it is necessary to return to lower levels where the content of these symbols can be perceived. But with this content, these symbols belong to a relational system which cannot be axiomatized because it cannot have direct liaison with logic.

2. Just as a child only learns his native language by applying grammatical rules (which are deduced from current usage), he only learns mathematics by applying mathematical rules. These rules only become firm, that is, become explicit, when one questions oneself about activities displayed at a lower level. It is in this way that all mathematical rules are formed, even the rules of formal logic. The application of rules is important, but the rule of application resides above all in the exploration of new domains bordering those where the rules and laws have been developed.

3. Two or more people can understand each other in a specified area of thought when they use a language in which they experience the same relations between the linguistic signs. The certainty of mathematics is based on the infallible way in which mathematical language can be used. The "mathematician at any price" can be happy with this: LANGUAGE is everything for him and he hardly cares what a symbol represents for others. (Just think of the point-line duality in the projective plane!) There is no problem from the algorithmic point of view. But if one is also concerned with knowing if agreement will still occur when the field of investigation is broadened, it is desirable to examine whether the symbols used by the questioner have a common base. It will not suffice then to learn the linguistic symbols and their liaisons, but it will be necessary to start with the same material at the lower level and to see if one succeeds, starting from there, in developing the same domains of higher level symbols.

Description of a Geometry Course

The first part of a geometry course ought to allow the attainment of the first level of thought, which we will call the *aspect of geometry*. The aim of teaching is as follows; geometric figures such as cubes, squares, rhombuses, rectangles, circles, etc., should become bearers of their properties. A rhombus is no longer recognized by its appearance, but, for example, by the fact that the sides are equal or that the diagonals are perpendicular and bisect each other, or these two properties together.

relational system in which new operations are impossible. One finds an example of such an error in the teaching of fractions in Holland. In this instruction, a verbal relational system is established. For most of the students, operations with fractions are completely incomprehensible. If in teaching, the teachers only recognized that the relational system of the students is more valuable than that of the teachers!

The heart of the idea of levels of thought lies in the statement that in each scientific discipline, it is possible to think and to reasons at different levels, and that this reasoning calls for different languages. These languages sometimes use the same linguistic symbols, but these symbols do not have the same meaning in such a case, and are connected in a different way to other linguistic symbols. This situation is an obstacle to the exchange of views which goes on between teacher and student about the subject matter being taught. It can perhaps be considered the fundamental problem of didactics.

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